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A Note on Incomplete Integrals of Cylindrical Functions

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Representations for incomplete Lipschitz-Hankel integrals of cylindrical functions are given in closed form by using Kampé de Fériet double hypergeometric functions. In addition, reduction formulas for the Kampé de Fériet functions associated with these integrals are provided for some cases.

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A NOTE ON INCOMPLETE INTEGRALS OF CYLINDRICAL FUNCTIONS

INTRODUCTION

The class of cylindrical functions C includes Bessel functions of the first kind J, modified Bessel functions I, Bessel functions of the second kind or Neumann functions Y (or N), Bessel functions of imaginary argument or MacDonald functions K, and Bessel functions of the third kind that include Hankel functions of the first and second kind, $H^{(1)}$ and $H^{(2)}$.

The general incomplete Lipschitz-Hankel integral of cylindrical functions $C_\nu(z)$ is defined as the function of two complex variables:

$$C_{e_{\mu,\nu}}(a, z) \equiv \int_0^z e^{at} t^\mu C_\nu(t) dt . \tag{1}$$

Here the symbol e denotes the presence of the exponential function and μ, ν may be complex. Analogously, we define integrals that contain the functions sin (at) and cos (at) in place of exp (at):

$$C_{s_{\mu,\nu}}(a, z) \equiv \int_0^z \sin (at)\, t^\mu C_\nu(t) dt \tag{2}$$

$$C_{c_{\mu,\nu}}(a, z) \equiv \int_0^z \cos (at)\, t^\mu C_\nu(t) dt . \tag{3}$$

To assure convergence of $C_{e_{\mu,\nu}}(a, z)$ and $C_{c_{\mu,\nu}}(a, z)$, it is necessary that Re $(\mu + 1) > |\text{Re}\, \nu|$ when $C = K, Y, H^{(1)}, H^{(2)}$; Re $(1 + \mu + \nu) > 0$ when $C = I, J$. When $\mu = \nu$, we define, for example, $C_{e_{\mu,\mu}} \equiv C_{e_\mu}$ where for convergence Re $\mu > -1/2$ for all C.

Integrals of the type given by Eqs. (1) to (3) occur very often in applied mathematics. Agrest and Maksimov [1] have found representations for $C_{e_\mu}(a, z)$, $C_{s_\mu}(a, z)$, and $C_{c_\mu}(a, z)$ using incomplete cylindrical functions. In this report we give representations for $C_{e_{\mu,\nu}}(a, z)$, $C_{s_{\mu,\nu}}(a, z)$, and $C_{c_{\mu,\nu}}(a, z)$ using only the Kampé de Fériet double hypergeometric functions $F^{0:2;1}_{2:1;0}[x, y]$.

PRELIMINARY RESULTS AND DEFINITIONS

To begin, we summarize some results that are found in Ref. 2, p. 85: Let a and b be arbitrary constants,

$$\mathbf{F}_\nu(z) \equiv aI_\nu(z) + be^{i\nu\pi}K_\nu(z)$$

$$\mathbf{G}_\nu(z) \equiv aJ_\nu(z) + bY_\nu(z)$$

$$\alpha \equiv \begin{cases} i: \mathbf{H} = \mathbf{F} \\ 1: \mathbf{H} = \mathbf{G} \end{cases} \qquad \beta \equiv \begin{cases} 1: \mathbf{H} = \mathbf{F} \\ 0: \mathbf{H} = \mathbf{G} \end{cases}$$

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Then

$$\int_0^z t^\mu \mathbf{H}_\nu(t)dt = e^{-\frac{\pi}{2}i\beta\mu}[(\mu+\nu-1)z\mathbf{H}_\nu(z)s_{\mu-1,\nu-1}(\alpha z) + (2\beta-1)\alpha z\mathbf{H}_{\nu-1}(z)s_{\mu,\nu}(\alpha z)], \quad (4)$$

where the Lommel functions $s_{\mu,\nu}$ are given by

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left[1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};\frac{-z^2}{4}\right]. \quad (5)$$

Now defining

$$\xi \equiv \begin{cases} 1: & C = I, K \\ -1: & C = H, J, Y \end{cases} \qquad \eta \equiv \begin{cases} 1: & C = K \\ -1: & C = H, I, J, Y \end{cases}$$

we may deduce from Eqs. (4) and (5) the result

$$\int_0^z t^\mu C_\nu(t)\,dt = \frac{z^{\mu+1}}{\mu-\nu+1}\left\{C_\nu(z)\,{}_1F_2\left[1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+1}{2};\frac{\xi z^2}{4}\right]\right.$$

$$\left. + \frac{\eta z C_{\nu-1}(z)}{\mu+\nu+1}\,{}_1F_2\left[1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};\frac{\xi z^2}{4}\right]\right\}. \quad (6)$$

We define the Kampé de Fériet double hypergeometric functions L and Q and give associated generating relations [3, 4]:

$$L[\alpha,\beta;\gamma,\delta;x,y] \equiv F^{0:1;1}_{2:0;0}\left[\begin{array}{l} \text{---}:\ \alpha;\ \beta; \\ \\ \gamma,\delta:\ \text{---};\ \text{---}; \end{array}\ x, y\right], \quad |x| < \infty, \quad |y| < \infty$$

$$Q[\alpha,\beta,\gamma;\mu,\nu,\lambda;x,y] \equiv F^{0:2;1}_{2:1;0}\left[\begin{array}{l} \text{---}:\ \alpha,\beta;\ \gamma; \\ \\ \mu,\nu:\ \lambda;\ \text{---}; \end{array}\ x, y\right], \quad |x| < \infty, \quad |y| < \infty$$

$$L[\alpha,\beta;\gamma,\delta;x,y] = \sum_{m=0}^\infty \frac{(\alpha)_m}{(\gamma)_m(\delta)_m}\frac{x^m}{m!}\,{}_1F_2[\beta;m+\gamma,m+\delta;y]$$

$$Q[\alpha,\beta,\gamma;\mu,\nu,\lambda;x,y] = \sum_{m=0}^\infty \frac{(\alpha)_m(\beta)_m}{(\mu)_m(\nu)_m(\lambda)_m}\frac{x^m}{m!}\,{}_1F_2[\gamma;m+\mu,m+\nu;y]. \quad (7)$$

It is easy to see that the function L is a special case of Q:

$$Q[\alpha,\lambda,\beta;\gamma,\delta,\lambda;x,y] = L[\alpha,\beta;\gamma,\delta;x,y].$$

For brevity we define the parameter lists

$$A_1(\mu, \nu) \equiv \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$A_2(\mu, \nu) \equiv \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$B_1(\mu, \nu) \equiv \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$B_2(\mu, \nu) \equiv \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 4}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$D_1(\mu) \equiv \frac{1}{2} + \mu, 1; \frac{1}{2} + \mu, \frac{3}{2}$$

$$D_2(\mu) \equiv \frac{1}{2} + \mu, 1; \frac{3}{2} + \mu, \frac{3}{2}$$

$$E_1(\mu, \nu) \equiv \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{2 + \mu + \nu}{2}, \frac{3 + \mu + \nu}{2}, 1 + \nu$$

$$E_2(\mu, \nu) \equiv \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{3 + \mu + \nu}{2}, \frac{4 + \mu + \nu}{2}, 1 + \nu$$

$$F_1(\mu) \equiv \frac{1}{2} + \mu, 1; 1 + \mu, \frac{3}{2} + \mu$$

$$F_2(\mu) \equiv \frac{1}{2} + \mu, 1; 2 + \mu, \frac{3}{2} + \mu$$

REPRESENTATIONS FOR $C_{e_{\mu,\nu}}(a, z)$, $C_{s_{\mu,\nu}}(a, z)$, $C_{c_{\mu,\nu}}(a, z)$

Substituting the Maclaurin series for $\exp(at)$ in Eq. (1) and splitting into even and odd terms we obtain on integrating term by term

$$C_{e_{\mu,\nu}}(a, z) = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \int_0^z t^{\mu + 2n} C_\nu(t) dt + \sum_{n=0}^{\infty} \frac{a^{1 + 2n}}{(1 + 2n)!} \int_0^z t^{1 + \mu + 2n} C_\nu(t) dt .$$

Then using Eq. (6) and the generating relation Eq. (7) we obtain after a tedious but straightforward computation the principal result of this note

$$C_{e_{\mu,\nu}}(a, z) = z^{1+\mu} C_\nu(z) \left\{ \frac{1}{\mu-\nu+1} Q\left[A_1; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{az}{\mu-\nu+2} Q\left[B_1; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}$$

$$+ \eta z^{2+\mu} C_{\nu-1}(z) \left\{ \frac{1}{(\mu+\nu+1)(\mu-\nu+1)} Q\left[A_2; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right.$$

$$\left. + \frac{az}{(\mu+\nu+2)(\mu-\nu+2)} Q\left[B_2; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}. \tag{8}$$

Since

$$C_{s_{\mu,\nu}}(a, z) = \frac{1}{2i} \left\{ C_{e_{\mu,\nu}}(ia, z) - C_{e_{\mu,\nu}}(-ia, z) \right\}$$

$$C_{c_{\mu,\nu}}(a, z) = \frac{1}{2} \left\{ C_{e_{\mu,\nu}}(ia, z) + C_{e_{\mu,\nu}}(-ia, z) \right\}$$

we may write

$$C_{s_{\mu,\nu}}(a, z) = \frac{az^{2+\mu}}{\mu-\nu+2} \left\{ C_\nu(z) Q\left[B_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right.$$

$$\left. + \frac{\eta z}{\mu+\nu+2} C_{\nu-1}(z) Q\left[B_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\} \tag{9}$$

$$C_{c_{\mu,\nu}}(a, z) = \frac{z^{1+\mu}}{\mu-\nu+1} \left\{ C_\nu(z) Q\left[A_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right.$$

$$\left. + \frac{\eta z}{\mu+\nu+1} C_{\nu-1}(z) Q\left[A_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}. \tag{10}$$

For $\mu = \nu$, Eqs. (8) to (10) reduce to

$$C_{e_\mu}(a, z) = z^{1+\mu} C_\mu(z) \left\{ L\left[D_1; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{az}{2} Q\left[B_1(\mu, \mu); \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}$$

$$+ \eta z^{2+\mu} C_{\mu-1}(z) \left\{ \frac{1}{1+2\mu} L\left[D_2; \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{az}{4(1+\mu)} Q\left[B_2(\mu, \mu); \frac{a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\} \tag{11}$$

$$C_{s_\mu}(a, z) = \frac{1}{2} az^{2+\mu} \left\{ C_\mu(z) Q\left[B_1(\mu, \mu); \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right.$$

$$\left. + \frac{\eta z}{2(1+\mu)} C_{\mu-1}(z) Q\left[B_2(\mu, \mu); \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\} \tag{12}$$

$$C_{c_\mu}(a, z) = z^{1+\mu}\left\{C_\mu(z) L\left[D_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{\eta z}{1+2\mu} C_{\mu-1}(z) L\left[D_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}. \quad (13)$$

Defining $J^+ \equiv J$, $J^- \equiv I$, it is interesting to note that we may also write [6]

$$J^{\pm}_{e_{\mu,\nu}}(a, z) = \frac{z^{1+\mu+\nu} e^{az}}{2^\nu (1+\mu+\nu)\Gamma(1+\nu)}\left\{Q\left[E_1; \frac{\mp z^2}{4}, \frac{a^2z^2}{4}\right] - \frac{az}{2+\mu+\nu} Q\left[E_2; \frac{\mp z^2}{4}, \frac{a^2z^2}{4}\right]\right\}$$

$$J^{\pm}_{e_\mu}(a, z) = \frac{z(z^2/2)^\mu e^{az}}{(1+2\mu)\Gamma(1+\mu)}\left\{L\left[F_1; \frac{\mp z^2}{4}, \frac{a^2z^2}{4}\right] - \frac{az}{2(1+\mu)} L\left[F_2; \frac{\mp z^2}{4}, \frac{a^2z^2}{4}\right]\right\}. \quad (14)$$

Here the Bessel functions $J^{\pm}_\nu$ do not appear.

REDUCTION FORMULAS FOR L AND Q

Many special cases of Eqs. (11) to (14) may be obtained in one form or another, provided we know a reduction formula for either L or Q. We summarize some known relevant reduction formulas [3-6]:

$$L[\alpha, \beta; \gamma, \delta; z, z] = {}_1F_2[\alpha + \beta; \gamma, \delta; z]$$

$$L\left[D_2; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{\sinh z}{z}$$

$$L\left[D_1; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{2\mu}{1+2\mu}\frac{\sinh z}{z} + \frac{\cosh z}{1+2\mu}$$

$$Q\left[B_2(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{1+\mu}{1+2\mu}\frac{4}{z^2}\left\{\cosh z - \left(\frac{2}{z}\right)^\mu \Gamma(1+\mu) I_\mu(z)\right\}$$

$$Q\left[B_1(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{2}{1+2\mu}\frac{1}{z}\left\{2\mu\frac{\cosh z}{z} + \sinh z - \left(\frac{2}{z}\right)^\mu \Gamma(1+\mu) I_{\mu-1}(z)\right\}.$$

Other properties and reduction formulas for L and Q are found in Refs. 3-6.

APPLICATIONS

Of interest in applications are the functions $J_{e_0}(a, z)$, $I_{e_0}(a, z)$, $Y_{e_0}(a, z)$, and $K_{e_0}(a, z)$. $J_{e_0}(a, z)$ and $Y_{e_0}(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_{e_0}(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study ot resonant absorption in media with finite dimensions [1, p. 195]. $K_{e_0}(a, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of

interest in the study of the scattered coherent reflected field from the sea surface [7, 8]. Since the functions $C_{e_0}(a, z)$ are of some importance, by using Eq. (11) and defining

$$L_1(x, y) \equiv L\left[\frac{1}{2}, 1; \frac{1}{2}, \frac{3}{2}; x, y\right]$$

$$L_0(x, y) \equiv L\left[\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; x, y\right]$$

$$Q_1(x, y) \equiv Q\left[1, 1, 1; 1, 2, \frac{3}{2}; x, y\right]$$

$$Q_0(x, y) \equiv Q\left[1, 1, 1; 2, 2, \frac{3}{2}; x, y\right]$$

we obtain

$$K_{e_0}(a, z) = zK_0(z)\left\{L_1\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{2}Q_1\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$+ z^2K_1(z)\left\{L_0\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{4}Q_0\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$Y_{e_0}(a, z) = zY_0(z)\left\{L_1\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right] + \frac{az}{2}Q_1\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right]\right\}$$

$$+ z^2Y_1(z)\left\{L_0\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right] + \frac{az}{4}Q_0\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right]\right\}$$

$$J_{e_0}(a, z) = zJ_0(z)\left\{L_1\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right] + \frac{az}{2}Q_1\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right]\right\}$$

$$+ z^2J_1(z)\left\{L_0\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right] + \frac{az}{4}Q_0\left[\frac{a^2z^2}{4}, \frac{-z^2}{4}\right]\right\}$$

$$I_{e_0}(a, z) = zI_0(z)\left\{L_1\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{2}Q_1\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$- z^2I_1(z)\left\{L_0\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{4}Q_0\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}.$$

The equations for $H_{e_0}^{(1)}$ and $H_{e_0}^{(2)}$ are the same as those for Y_{e_0} or J_{e_0} with Y or J replaced by $H^{(1)}$ or $H^{(2)}$. Further, from Eq. (14) we have

$$J_{e_0}(a, z) = ze^{az}\left\{L\left[1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2z^2}{4}, \frac{-z^2}{4}\right] - \frac{az}{2} L\left[1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2z^2}{4}, \frac{-z^2}{4}\right]\right\}$$

$$I_{e_0}(a, z) = ze^{az}\left\{L\left[1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2z^2}{4}, \frac{z^2}{4}\right] - \frac{az}{2} L\left[1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}.$$

Here we have used the properties of L that

$$L[\alpha, \beta; \gamma, \delta; x, y] = L[\alpha, \beta; \delta, \gamma; x, y] = L[\beta, \alpha; \gamma, \delta; y, x].$$

The latter results for $C_{e_0}(a, z)$ should prove useful in numerical computation of these functions.

SUMMARY

Representations for incomplete Lipschitz-Hankel integrals of cylindrical functions using only the Kampé de Fériet functions in two variables $F_{2:1;0}^{0:2;1}[x, y]$ are given. In addition, known relevant reduction formulas for these functions are provided.

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